Lecture 15 30-10-18

Recall: we considered linear system of 1^{st} order eqt:

$$\vec{y}'(t) - P(t)\vec{y}(t) + r(t)$$

where $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $P(t) = \begin{pmatrix} P_{11}(t), \dots, P_{1n}(t) \\ \vdots \quad \ddots \quad \vdots \\ P_{n1}(t) \dots\dots P_{nn}(t) \end{pmatrix}$ $r(t) = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}$

Thm (Existence and unique for linear equation)
For system of 1^{st} order linear equation. if $P_{11}(t) \dots P_{1n}(t)$, $\dots\dots$, $P_{n1}(t), \dots, P_{nn}(t), r_1(t), \dots, r_n(t)$ continuous on I
then $\exists!$ solution $(y_1, \dots, y_n)^t = \vec{y}(t)$ on whole I.
for the IVP.

Prop: (superposition)

- Let $\vec{y}_1(t)$ satisfy $\vec{y}_1'(t) = P(t)\vec{y}_1(t) + \vec{r}_1(t)$
  $\vec{y}_2(t)$ satisfy $\vec{y}_2'(t) = P(t)\vec{y}_2(t) + \vec{r}_2(t)$.

then if we let $\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t)$
satisfy: $\vec{y}'(t) = P(t)\vec{y}(t) + c_1\vec{r}_1(t) + c_2\vec{r}_2(t)$

- In particular, if $\vec{r}_1 = \vec{r}_2 = \vec{0}$, then
  $\vec{y}'(t) = P(t)\vec{y}(t)$.

Cor: If we let $\mathcal{S} = \left\{ \vec{y} = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \middle| \vec{y}'(t) = P(t)\vec{y}(t) \right\}$

then $\mathcal{S}$ is a $\mathbb{R}$-vector space ($\mathbb{C}$-vector space
if $P(t)$ is $\mathbb{C}$-valued and if we look for $\mathbb{C}$-valued $\vec{y}(t)$)

§ Linear algebra review:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} \quad \text{for } a_{ij} \in \mathbb{R} \text{ or } \mathbb{C}$$

$m \times n$ matrix

System of Linear equation:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \iff A\vec{x} = \vec{b} \text{ in terms of matrix.}$$

i.e.
- We are interested in solving linear equation $A\vec{x} = \vec{b}$
- we will focus on the case for $m = n$.

§ Method of elementary row operation:

Eg.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned} \iff \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

row operation:

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_2]{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{4}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$\rightsquigarrow$ solution $x_1 = 2$, $x_2 = -1$, $x_3 = 1$.

Row operation for inverse:

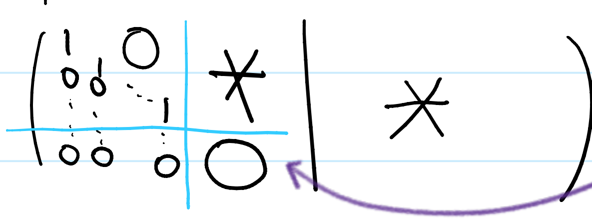
Q: Try to find inverse of $\begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - 4R_2]{R_1 \mapsto R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \mapsto -\frac{1}{5}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & -2/5 & 1/5 \end{array}\right) \xrightarrow[R_1 \mapsto R_1 - \frac{3}{2}R_3]{R_2 \mapsto R_2 - \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & -1/10 & 3/10 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right)$$

Rk: If the matrix A is NOT invertible, we will get



Cannot proceed anymore.

Def: • Given $A = (a_{ij})_{\substack{1\le i\le n \\ 1\le j\le n}}$ an $n\times n$ $\mathbb{C}$-matrix

we call $r \in \mathbb{C}$ an eigenvalue of A if we have

$$A\vec{v} = r\vec{v} \quad \text{for some } \vec{v} \neq \vec{0}.$$

• Let $P_A(x) := \det(A - xI)$ be the degree n polynomial which is called characteristic polynomial.

fact: $P_A(r) = 0$ if r is an eigenvalue of A

Def: r eigenvalue of A, we call $\ker(A - rI) := E_r$ the eigen-space w.r.t. r, and any $v \in E_r$ an eigenvector w.r.t. r

Def: Geometric multiplicity of eig. value $r := \dim_{\mathbb{C}}(\ker(A - rI))$

algebraic " of " " $r :=$ mult of the root r w.r.t the polynomial $P_A(x)$.

fact: 1. Geometric mult. of $r \leq$ algebraic multiplicity of r

2. A diagonalizable, meaning that $\exists\, Q$ invertible $\mathbb{C}$-matrix

$$Q^{-1}AQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \longleftrightarrow \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

eigenvalues

$\Longleftrightarrow$ geometric mult. of any r = algebraic mult of r.

E.g. $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$, Solve for eigenvalue and eigenspaces.

characteristic polynomial:

$$P_A(x) = \det(A - xI) = \det\begin{pmatrix} 1-x & 2 & 1 \\ 1 & -1-x & 1 \\ 2 & 0 & 1-x \end{pmatrix}$$

$$= -(x-3)(x+1)^2.$$

For $\lambda_1 = -1$: we have $A + I = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$

solve: $(A+I)(\vec{v}) = 0$.

Row operation: $\left(\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right)$

$\xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$

$\xrightarrow[R_1 \mapsto R_1 - R_2]{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}$$

hence geometric multiplicity of -1 is 1.

< algebraic " " -1.

(So it is NOT diagonalizable).

For $r = 3$:

$$A - 3I = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -4 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$ and we perform

row operation again to solve eigenvector $\begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}$.

Eg. (complex eigenvalues).

$$A = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}, \quad A - xI = \begin{pmatrix} -3-x & -2 \\ 4 & 1-x \end{pmatrix}$$

characteristic poly: $p_A(x) = -(3+x)(1-x) + 8$

$$= x^2 + 2x + 5.$$

$\lambda_1 = -1+2i$, $\lambda_2 = -1-2i$ are two eigenvalue.

For $\lambda_1 = -1+2i$:

$$A - \lambda_1 I = \begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} = \begin{pmatrix} (1+i)(-2) & -2 \\ (1+i)(2-2i) & 2-2i \end{pmatrix}$$

$\therefore$ corresponding eigenvector $v = \begin{pmatrix} -1 \\ (1+i) \end{pmatrix}$

For A being real-valued matrix:

$Av = \lambda_1 v$ taking conjugate gives $\overline{Av} = \overline{\lambda_1 v}$

$\Longrightarrow A\overline{v} = \underbrace{\overline{\lambda_1}}_{=\lambda_2}\, \vec{v}$ $\therefore$ $\underbrace{\vec{v}}_{=\begin{pmatrix} -1 \\ 1-i \end{pmatrix}}$ eigenvector w.r.t. λ_2.

Writing $Q^{-1}AQ = \text{diag}(\lambda_1, \lambda_2)$:

Let $Q = \begin{pmatrix} -1 & -1 \\ 1+i & 1-i \end{pmatrix}$, then we have

$$AQ = \begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix} = Q \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

which is the desired Q for diagonalizing A.

In general: If A is diagonalizable, then we let $\lambda_1, \dots, \lambda_n$ eigenvalues of A counting multiplicity with a basis of eigenvector $\begin{pmatrix} | \\ v_1 \\ | \end{pmatrix}, \dots, \begin{pmatrix} | \\ v_n \\ | \end{pmatrix}$

then we can take $Q = \begin{pmatrix} | & | & & | \\ v_1, & v_2, & \dots & v_n \\ | & | & & | \end{pmatrix}$ for diagonalizing A.

Def: Let $A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}$ function of t defining on an interval I

we let $\dfrac{dA}{dt} = \begin{pmatrix} a'_{11}(t) & \cdots & a'_{1n}(t) \\ \vdots & & \vdots \\ a'_{m1}(t) & \cdots & a'_{mn}(t) \end{pmatrix}$

and similarily for vectors.

fact:

- $$\boxed{\frac{d(AB)}{dt} = \frac{dA}{dt}B + A\frac{dB}{dt}}$$

  be careful about the order of multiplication.

$\boxed{☆: \text{ For matrix, } AB \neq BA.}$

- If $A(t)$ invertible for all t, then if we let $B(t) = A^{-1}(t)$

$$AB = id \Rightarrow \frac{dA}{dt}B + A\frac{dB}{dt} = 0$$

$$\Rightarrow \boxed{\frac{d(A^{-1})}{dt} = -A^{-1}\left(\frac{dA}{dt}\right)A^{-1}}$$

Prop: $A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}$, then we have

$$\frac{d}{dt}\det(A(t)) = \det\begin{pmatrix} \frac{da_{11}}{dt} & \cdots & \frac{da_{1n}}{dt} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \frac{da_{21}}{dt} & \cdots & \frac{da_{2n}}{dt} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$+ \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \frac{da_{n1}}{dt} & \cdots & \frac{da_{nn}}{dt} \end{pmatrix}$$

also using the fact $\det(A) = \det(A^T)$

$$\frac{d}{dt}(\det(A(t))) = \det\begin{pmatrix} \frac{da_{11}}{dt} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \frac{da_{n1}}{dt} & \cdots & a_{nn} \end{pmatrix} + \cdots + \det\begin{pmatrix} a_{11} & \cdots & \frac{da_{1n}}{dt} \\ \vdots & & \vdots \\ a_{n1} & \cdots & \frac{da_{nn}}{dt} \end{pmatrix}$$

§ Homogeneous system of linear equation:

We consider: $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ with $P(t) = \begin{pmatrix} P_{11}(t) & \cdots & P_{1n}(t) \\ \vdots & \ddots & \vdots \\ P_{n1}(t) & \cdots & P_{nn}(t) \end{pmatrix}$

and $\vec{y}\,'(t) = P(t) \cdot \vec{y}(t)$ --- $(*)$.

Def: Let $\vec{y}_1, \ldots, \vec{y}_n$ be n solution to $(*)$, we define the matrix

$$\mathbb{X}(t) := \begin{pmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_n \\ | & | & & | \end{pmatrix}$$

and let the Wronskian $W(\vec{y}_1, \ldots, \vec{y}_n)(t) = \det(\mathbb{X}(t))$.

Thm: $\vec{y}_1, \ldots, \vec{y}_n$ solution to $*$, we have TFAE

i) $\vec{y}_1, \ldots, \vec{y}_n$ linearly independent

ii) $\mathrm{Span}(\vec{y}_1, \ldots, \vec{y}_n) := \{c_1\vec{y}_1 + \cdots + c_n\vec{y}_n \mid c_i \in \mathbb{R}\}$ equal to the space of solution of $*$

iii) $W(\vec{y}_1, \ldots, \vec{y}_n)(t_0) \neq 0$ at $t_0 \in I$.

Pf: We prove the equivalence between

1) $\vec{y}_1, \dots, \vec{y}_n$ is a basis for $\mathcal{S}$

2) $W(\vec{y}_1, \dots, \vec{y}_n)(t_0) \neq 0$

1) $\Longrightarrow$ 2)

- By existence result, for each $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th}$

we have $z_i \in \mathcal{S}$ s.t. $z_i(t_0) = e_i$.

- By $\vec{y}_1, \dots, \vec{y}_n$ is a basis, we have

$$c_{i1}\vec{y}_1 + \dots + c_{in}\vec{y}_n = z_i$$

, which in terms of matrix gives us (by evaluating at t_0)

$$\begin{pmatrix} | & | & & | \\ y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\Longrightarrow \det(\Phi(t_0)) \neq 0$.

2) $\Longrightarrow$ 1).

- $\underline{\mathrm{Span}(\vec{y}_1, \dots, \vec{y}_n) = \mathcal{S}}$:

Let z be any solution in $\mathcal{S}$, we can solve the equation

$$\begin{pmatrix} | & & | \\ y_1(t_0) & \dots & y_n(t_0) \\ | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} | \\ z(t_0) \\ | \end{pmatrix}$$

Therefore by uniqueness $\Longrightarrow$ $c_1\vec{y}_1 + \cdots + c_n\vec{y}_n = Z$.

Linearly independent :

if $c_1\vec{y}_1 + \cdots + c_n\vec{y}_n = 0$ as a function

evaluating at t_0:

$$c_1\vec{y}(t_0) + \cdots + c_n\vec{y}_n(t_0) = 0 \text{ as a vector in } \mathbb{R}^n$$

$\det\begin{pmatrix} | & & | \\ y_1(t_0) & \cdots & y_n(t_0) \\ | & & | \end{pmatrix} \neq 0 \Rightarrow \vec{y}_1(t_0), \ldots, \vec{y}_n(t_0)$ linearly independent as vector in $\mathbb{R}^n$.

$\Longrightarrow$ $c_1 = \cdots = c_n = 0$

Def: $\vec{y}_1, \ldots, \vec{y}_n \in \mathcal{S}$ is called a fundamental set of solution if we have $W(\vec{y}_1, \ldots, \vec{y}_n)(t_0) \neq 0$ for some $t_0 \in I$. $X(y_1, \ldots, y_n)(t)$ is called the fundamental matrix.

Prop: equation $(*)$ always has a fundamental set of solution $\vec{y}_1, \ldots, \vec{y}_n$.

Pf: By existence: we take y_i be a solution such that $y_i(t_0) = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← i-th

$\rightsquigarrow$ $W(\vec{y}_1, \ldots, \vec{y}_n)(t_0) = 1$.

Thm: **(Liouville's formula)** Let $\vec{y}_1, \dots, \vec{y}_n$ be solution to $(*)$ then we have

$$W(\vec{y}_1, \dots, \vec{y}_n)(t) = C \exp\left(\int tr(P(t))\, dt \right)$$

Pf:

$$W(y_1, \dots, y_n)(t) = \det \Phi(t) = \det \begin{array}{c} \begin{array}{ccccc} & = \vec{y}_2 & & = \vec{y}_n \end{array} \\ \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} \end{array}$$

and using previous formula we have

$$W'(y_1 \dots y_n)(t) = \det \begin{pmatrix} | & | & & | \\ y_1' & y_2 & \cdots & y_n \\ | & | & & | \end{pmatrix} + \det \begin{pmatrix} | & | & & | \\ y_1 & y_2' & \cdots & y_n \\ | & | & & | \end{pmatrix}$$

$$+ \det \begin{pmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_n' \\ | & | & & | \end{pmatrix}$$

We let a matrix $M(t)$ s.t.

$$(M(t))_{ij} = \det \begin{array}{cc} \begin{array}{cccccc} & & & i & & \end{array} & \\ \begin{pmatrix} y_{11} & y_{12} & \cdots & 0 & & y_{1n} \\ \vdots & | & & 0 & & \vdots \\ 0 & \cdots & 0\ 0 & 1 & 0 & 0 \\ \vdots & & & 0 & & \vdots \\ y_{1n} & y_{n2} & & 0 & & y_{nn} \end{pmatrix} & \begin{array}{c} \\ \\ j \\ \\ \\ \end{array} \end{array}$$

Then we have $\Phi(t) M(t) = \begin{pmatrix} \det \Phi & & 0 \\ & \ddots & \\ 0 & & \det \Phi \end{pmatrix}$

by the Cramer's rule.

Therefore we can write: $\det \begin{pmatrix} | & | & & | \\ y_1' & y_2 & \cdots & y_n \\ | & | & & | \end{pmatrix}$

$$= y_{11}' M_{11} + y_{12}' M_{12} + \cdots + y_{1n}' M_{1n}.$$

And similarly:

$$\left(M(t)X'(t)\right)$$
$$\|$$
$$\det\begin{pmatrix} | & & | & & | \\ y_1 & \cdots & y'_k & \cdots & y_n \\ | & & | & & | \end{pmatrix} = y'_{k1}M_{k1} + \cdots + y'_{k1}M_{k1}$$

$\therefore$ summing up we get.

$$W'(y_1,\dots,y_n)(t) = \left(M(t)\,X'(t)\right)_{11} + \cdots + \left(M(t)X'(t)\right)_{nn}.$$
$$= \operatorname{tr}\left(M(t)\,X'(t)\right).$$

$$X'(t) = P(t)\,X(t).$$

$$\Rightarrow \quad W'(t) = \operatorname{tr}\left(M(t)\,P(t)\,X(t)\right) = \operatorname{tr}\left(X(t)M(t)P(t)\right)$$
$$= \det(X(t))\,\operatorname{tr}(P(t)).$$
$$= W(t)\,\operatorname{tr}(P(t)) \qquad \square$$